

# Analytic Study for the String Theory Landscapes via Matrix Models

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We demonstrate a first-principle analysis of the string theory landscapes in the framework of non-critical string/matrix models. In particular, we discuss non-perturbative instability, decay rate and the true vacuum of perturbative string theories. As a simple example, we argue that the perturbative string vacuum of pure gravity is stable; but that of Yang-Lee edge singularity is *inescapably* a false vacuum. Surprisingly, most of perturbative minimal string vacua are unstable, and their true vacuum mostly does not suffer from non-perturbative ambiguity. Importantly, we observe that the instability of these tachyon-less closed string theories is caused by *ghost D-instantons* (or ghost ZZ-branes), the existence of which is determined only by non-perturbative completion of string theory.

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## I. ANALYTIC ASPECTS OF THE STRING THEORY LANDSCAPE

The string theory landscape is a space of vacua in string theory, which hopefully includes the standard model in four dimension. Despite of its importance, its progress is mainly in its statistical aspects [1]; and little is known about analytic structures of the space. By “analytic structures” we mean general interrelationship among distinct perturbative string-theory vacua. Therefore, appearance of vacua in the landscape, relative stability/decay rate of vacua and identification of the true vacuum are included. In this short letter, by use of the non-perturbative completion, we demonstrate a prescription to extract analytic structures of the string theory landscapes from perturbative string theory.

The free energy of perturbative string theory,  $\mathcal{F}(g)$ , is an asymptotic series and is calculated from world-sheet conformal field theory [2]:

$$\mathcal{F}(g) \simeq \sum_{n=0}^{\infty} g^{2n-2} \mathcal{F}_n + \sum_I \theta_I g^{\gamma_I} \exp \left[ \frac{1}{g} \sum_{n=0}^{\infty} g^n \mathcal{F}_n^{(I)} \right] + \mathcal{O}(\theta^2). \quad (1)$$

The nonperturbative corrections are usually provided by D-instantons, i.e. their leading contributions,  $\mathcal{F}_0^{(I)}$ , are identified as D-instanton action  $\mathcal{S}_I = -\frac{1}{g} \mathcal{F}_0^{(I)}$  [3, 4]. The over-all coefficient  $\theta_I$  for each instanton is called D-instanton fugacity [5–7], which has no corresponding worldsheet observable. Usually, we assume that the D-instanton action is positive:  $\mathcal{S}_I > 0$ . However, a negative-action partner of the instanton,  $\mathcal{S}_{I_{\text{gh}}} = -\mathcal{S}_I < 0$  has also been observed [8] in non-critical string theory [9]. They are then generally defined as ghost D-branes (or ghost D-instantons) in (non-)critical string theory [10]. However, such a D-brane was not seriously taken into account, since it contradicts with perturbation theory. Existence of the D-branes is discussed very recently mainly

in resurgent analysis [11–13] and it was found that these branes must be generally encoded in non-perturbatively completion of string theory.<sup>1</sup> In this letter, we shall see how these ghost D-instantons play a role in formulating “analytic structures” of the string theory landscape.

Since the actions of ghost D-instantons are negative (their masses are negative), they are no longer “corrections” to perturbation theory; they are rather *indication of non-perturbative instability of the perturbative vacuum* [12]. However, this is a cause of confusions, because “in principle, the ghost partner is defined for every D-brane, but it does not necessarily mean that the string theory is unstable.” In fact, it is non-trivial to know *which ghost D-instantons are allowed (or not allowed) to appear in the spectrum*. Naively, this information is given by physics of the D-instanton fugacity  $\{\theta_I\}_I$ . However, it is subtle to directly deal with  $\{\theta_I\}_I$  since they are coefficients of exponentially small corrections which are supposed to be negligible in asymptotic expansions (See e.g. [14]). Therefore, we should first grasp complete information of D-instanton fugacity. In the following, we shall see that, once one can control the information of D-instanton fugacity, one can quantitatively extract most of analytic aspects of the string theory landscapes, including *metastability, its decay rates and the true vacuum*.

*The completion and fugacity* It is known that perturbative amplitudes, including instanton corrections, in various solvable string theories are obtained by the in-

<sup>1</sup> It is shown that “multi instanton-ghost-instanton sectors” have discrepancy with worldsheet predictions in the sense of  $\mathcal{F}^{(n|m)} \neq \mathcal{F}^{(n-m|0)}$  [12, 13], and this is a main objection to identifying it as “ghost D-branes”. However we insist on using the terminology because, according to the free-fermion analysis [6, 15], multi ghost-instanton sectors  $\mathcal{F}^{(0|m)}$  are simply obtained by flipping the sign of the ZZ-brane boundary state operators in the multi-instanton sectors  $\mathcal{F}^{(m|0)}$  in all-order perturbation theory.

formation of spectral curves, especially with topological recursions [16, 17]. In particular, all-order asymptotic expansion of Eq. (1) is explicitly shown in [17], with fugacity remaining free parameters. Then, for completion of the non-perturbative information in the asymptotic expansion, there are mainly studied two ways to control fugacity: one is resurgent analysis (e.g. [11–14]); and the other is isomonodromy analysis (for mathematical developments on isomonodromy theory [18–20]; and with matrix models [21–24]). The former is based on connection formula (or Stokes phenomena) for analytic continuation of  $g$ ; and the latter is based on *Stokes phenomena of the Baker-Akhiezer (shortly BA) functions on the spectral curves*. Here we explore analytic aspects of the landscape from the latter approach.

## II. RIEMANN-HILBERT PROBLEM FOR THE BAKER-AKHIEZER FUNCTIONS

For a given spectral curve  $F(P, Q) = 0$  with a symplectic coordinate  $(P, Q)$ , we define the BA function as follows:

1) We define a function  $\varphi(\zeta)$  (called string-background) as

$$\varphi(\zeta) = \text{diag}(\varphi^{(j)}(\zeta)), \quad \varphi^{(j)}(\zeta) = \int^{\zeta} dP Q^{(j)}(P), \quad (2)$$

where  $\{Q^{(j)}(P)\}_{j=1}^k$  are branches of the algebraic equation, the number of which is an integer,  $k$ . The function  $\varphi(\zeta)$  is a rational function on the curve and, without loss of generality, it may have poles at  $\zeta = \infty$  and  $\zeta = \zeta_a$  ( $a = 1, 2, \dots, M-1$ ) in the following sense:

$$\begin{aligned} \varphi(\zeta) &\sim \sum_{n=1}^{r_0} \varphi_{-n} \lambda^n + \mathcal{O}\left(\frac{1}{\lambda}\right), \quad \zeta = \lambda^{\hat{p}_0} \rightarrow \infty, \\ \varphi(\zeta) &\sim \sum_{n=1}^{r_a} \frac{\varphi_{-n}(\zeta_a)}{\lambda^n} + \mathcal{O}(\lambda), \quad \zeta = \zeta_a + \lambda^{\hat{p}_a} \rightarrow \zeta_a, \end{aligned} \quad (3)$$

with  $a = 1, 2, \dots, M-1$ . Here  $\{\hat{p}_a\}_{a=0}^{M-1}$  are proper integers and  $\{r_a\}_{a=0}^{M-1}$  are the Poincaré indices. The BA function  $\Psi(\zeta)$  is then a  $k \times k$  matrix-valued sectional holomorphic function of  $\zeta \in \mathbb{C}^* \setminus \mathcal{K}$  as

$$\Psi(\zeta) = Z(\zeta) e^{\varphi(\zeta)} \prod_{a=0}^{M-1} (\zeta - \zeta_a)^{\nu_a / \hat{p}_a} \equiv Z(\zeta) e^{\tilde{\varphi}(\zeta)}, \quad (4)$$

where  $\mathcal{K}$  is a collection of connected line elements,  $\mathcal{K} = \bigcup_m \mathcal{K}_m$  equipped with a direction (e.g. Fig. 1 and Fig. 2), and  $\zeta_0 = 0$ . We often put  $\hat{p}_0 = \hat{p}$ ,  $r_0 = r$  and  $\nu_0 = -\nu$ .

Note that the line elements of the graph  $\mathcal{K}$  flow from the poles of  $\varphi(\zeta)$  (i.e. essential singularities of the BA function) and are drawn along anti-Stokes lines,  $\text{Re}[(\varphi^{(j)}(\zeta) - \varphi^{(l)}(\zeta))e^{-i\theta}] = 0$ , with a proper  $\theta$  so that the graph  $\mathcal{K}$  attaches to saddle points,  $\partial_{\zeta}(\varphi^{(j)}(\zeta) - \varphi^{(l)}(\zeta)) = 0$ .

2) For each segment of the graph,  $\mathcal{K}_m$ , a  $k \times k$  matrix  $S_m$  is assigned (which is called a *Stokes matrix*) and then the BA function has discontinuity along the segment,

$$\Psi(\zeta + \epsilon) = \Psi(\zeta - \epsilon) S_m, \quad \zeta \in \mathcal{K}_m, \quad (5)$$

where  $\epsilon$  directs to the left-hand side of the segment  $\mathcal{K}_m$ . In particular, at the poles of  $\varphi(\zeta)$ , there are a number of lines (as in Eq. (7)) and the Stokes matrices are defined so that the BA function has standard asymptotic expansion around them:

$$\begin{aligned} \Psi(\zeta) &\underset{\text{asym}}{\simeq} \left[ I_k + \sum_{n=1}^{\infty} \frac{Z_n}{\lambda^n} \right] e^{\tilde{\varphi}(\zeta)} \quad (\zeta = \lambda^{\hat{p}} \rightarrow \infty), \\ \Psi(\zeta) &\underset{\text{asym}}{\simeq} \left[ I_k + \sum_{n=1}^{\infty} \lambda^n Z_n(\zeta_a) \right] e^{\tilde{\varphi}(\zeta)} E_a \\ &\quad (\zeta = \zeta_a + \lambda^{\hat{p}_a} \rightarrow \zeta_a), \end{aligned} \quad (6)$$

with  $\det E_a \neq 0$  and  $a = 1, 2, \dots, M-1$ . Note that the expansion (6) does not depend on the direction of  $\zeta \rightarrow \zeta_0$ , and therefore this requires the standard form for the Stokes matrices (5).

Note that if one flips the direction of a line  $\mathcal{K}_m$  then the matrix is replaced by its inverse,  $S_m^{-1}$ . At a junction of lines, they satisfy a conservation equation:

$$\begin{array}{c} \begin{array}{c} 3 \quad 2 \quad 1 \\ \swarrow \quad \downarrow \quad \searrow \\ \vdots \quad \vdots \quad \vdots \end{array} \quad \Leftrightarrow \quad S_1 S_2 \cdots S_L = I_k. \end{array} \quad (7)$$

These are the basic algebra of the Stokes matrices. Obtaining explicit solutions to the algebra is a first non-trivial preparation for the Riemann-Hilbert (shortly RH) calculus, and some recent progress for general  $k$  and  $r$  can be found in [23, 24].

Importantly, the Stokes matrices  $S_m$  in Eq. (5) are independent from  $\zeta$  of  $\mathcal{K}_m$ , which means that the graph  $\mathcal{K}$  is topological and one can deform it continuously. In addition, we assume that the Stokes matrices are independent from the deformations of the leading Laurent coefficients in Eq. (3), i.e.  $\{\varphi_{-n}\}_{n=1}^{r_0}$  and  $\{\varphi_{-n}(\zeta_a)\}_{n=1}^{r_a} \overset{M-1}{a=1}$ . They are the isomonodromy deformations which guarantees the integrable hierarchy (e.g. KP/Toda hierarchy according to the spectral curves [25]) and string equations of the system, which means that perturbative results coincidence with that of the topological recursions. The partition function of matrix models is then given by the  $\tau$ -function of the integrable hierarchy (See e.g. [26]).

3) The sectional holomorphic function  $Z(\zeta)$  is uniquely fixed by giving the Stokes matrices of jump relations (5). In fact,  $Z(\zeta)$  is calculable by solving the Riemann-Hilbert integral equation (See e.g. [20]) around  $\zeta = \lambda^{\hat{p}} \rightarrow \infty$ :

$$\tilde{Z}(\lambda) = I_k + \int_{\mathcal{K}} \frac{d\xi}{2\pi i} \frac{\tilde{Z}(\xi - \epsilon)(G(\xi) - I_k)}{\xi - \lambda}, \quad (8)$$

along  $\mathcal{K}$ . Here  $\tilde{Z}(\lambda) \equiv Z(\lambda^{\hat{p}})$  and  $G(\lambda)$  is a sectional holomorphic function along  $\mathcal{K}$ , defined by  $G(\lambda) = e^{\tilde{\varphi}(\lambda^{\hat{p}})} S_m e^{-\tilde{\varphi}(\lambda^{\hat{p}})}$  ( $\lambda \in \mathcal{K}_m$ ;  $m = 0, 1, \dots$ ).

We should note that by this procedure one observes that fugacity is given by Stokes multipliers of  $\{S_m\}_m$  mostly related linearly (See e.g. [20]). Importantly this allows us to obtain the connection rules for analytic continuation of integrable flows (including string coupling  $g$ ). In this sense, the information of graph and matrices,  $\hat{\mathcal{K}} \equiv \bigcup_m (\mathcal{K}_m, S_m)$  has all the information of D-instanton fugacity of Eq. (1). We shall refer to  $\hat{\mathcal{K}}$  as *the Deift-Zhou (or DZ) network* [27] (by following the recent naming fashion [28]). This is how we control the fugacity.

4) In the RH approach, if one fixes the integrable flows  $\{\varphi_{-n}(\zeta_a)\}_{n=1}^{r_a} \stackrel{M}{\rightarrow}$  and Stokes matrices  $\{S_m\}_m$ , every information is determined. In particular, the BA function  $\Psi(\zeta)$  is not changed by any deformations of the spectral curve,  $F(P, Q) = 0 \rightarrow \tilde{F}(P, Q) = 0$  (i.e.  $\varphi(\zeta) \rightarrow \tilde{\varphi}(\zeta)$ ), such that the resulting string-background  $\tilde{\varphi}(\zeta)$  of Eq. (2) does not change the singular structure (3). In other words, this is just a matter of how divide the BA function  $\Psi(\zeta)$  into  $Z(\zeta)$  and  $\bar{\varphi}(\zeta)$ . In this sense, the RH approach can be interpreted as an (off-shell) background independent formulation of string theory [23].

By this fact, we define the string theory landscape  $\mathfrak{L}_{\text{str}}$  by the moduli space of spectral curves  $F(P, Q) = 0$  which preserves the pole structure (i.e. integrable flows) of  $\varphi(\zeta)$  in Eq. (3). Schematically, we define it as a set of string-background  $\varphi(\zeta)$ :

$$\mathfrak{L}_{\text{str}} = \{\varphi(\zeta); \text{keeping Eq. (3)}\}, \quad (9)$$

and the potential of the landscape is determined by the RH integral equation (8).

Note that the background independence of non-critical string theory was first explicitly shown in the topological recursions [17] i.e. within perturbation theory; however for the potential picture of landscape we need to know the non-perturbative completion of the string theory. In the following, as a first non-trivial example for the analytic aspects of the landscape, we shall show how metastability, decay rates and true vacuum are obtained with the information of fugacity/network which are controlled as above.

### III. CASES OF MINIMAL STRING THEORY

We now consider minimal string theory [9], as an example described by matrix models/spectral curves [29]. The spectral curve of  $(p, q)$  minimal string theory is given by

$$F(\zeta, Q) = T_p(Q/\beta\mu^{q/2p}) - T_q(\zeta/\sqrt{\mu}) = 0, \quad (10)$$

with Chebyshev polynomials of the first kind  $T_n(\cos \theta) = \cos n\theta$  [8, 30]. Therefore,  $\varphi(\zeta) (\equiv \varphi_{\text{mstr}}(\zeta))$  in this background is given as  $\varphi_{\text{mstr}}^{(j)}(\zeta) = \varphi_{\text{mstr}}^{(1)}(e^{-2\pi i \frac{j-1}{p}} \zeta)$  with  $\varphi_{\text{mstr}}^{(1)}(\zeta) = \beta\mu^{\frac{q+2}{4}} \int^{\zeta/\sqrt{\mu}} dx T_{q/p}(x)$ . That is,  $(p, q)$  minimal string theory is  $p \times p$  isomonodromy systems (i.e.  $k = p$ ) with only one essential singularity at  $\zeta = \infty$  of

Poincaré index  $r = p + q$  (we put  $\zeta = \lambda^p$  in Eq. (3)). We put the monodromy  $\nu_0$  as  $\nu_0 (= -\nu) = -\frac{p-1}{2}$ , and this background also preserves  $\mathbb{Z}_p$ -symmetry in the sense of [23]. Here following the discussion [23], we use the same notation.

Around the singularity  $\zeta \rightarrow \infty$ , there are  $2rp$  Stokes matrices  $\{S_n\}_{n=0}^{2rp-1}$  of  $p \times p$ , and their algebraic relations [20, 23] are expressed as

- $\mathbb{Z}_p$ -symmetry condition:  
 $S_{n+2r} = \Gamma^{-1} S_n \Gamma \quad (n = 0, 1, \dots, 2rp - 1),$
- Monodromy condition:  
 $S_0 S_1 \cdots S_{2rp-1} = e^{\pi i(p-1)} I_p,$
- Hermiticity condition:  
 $S_n^* = \Delta \Gamma S_{(2r-1)p-n}^{-1} \Gamma^{-1} \Delta \quad (n = 0, 1, \dots, 2rp - 1).$

with  $\Gamma = (\Gamma_{ij})_{1 \leq i, j \leq p} = (\delta_{j, i+1} + \delta_{i, p} \delta_{j, 1})_{1 \leq i, j \leq p}$  and  $\Delta = (\Delta_{ij})_{1 \leq i, j \leq p} = (\delta_{i+j, p+1})_{1 \leq i, j \leq p}$ . For components of the matrices  $\{S_n\}_{n=0}^{2rp-1}$ , one should consult [23].

In addition, we consider the cases related to matrix models. The corresponding conditions for the Stokes matrices are known as the multi-cut boundary condition [23]. In particular, in the case of  $(p, q)$  minimal string theory, the constraint is the same as  $p$ -cut critical points of the multi-cut matrix models [31–34].<sup>2</sup> A major difference from the previous cases [23, 24] is, however, that the Poincaré index is greater than the number of cuts:

$$r (= p + q) > k (= p), \quad (11)$$

which greatly simplify the quantum integrable structure of the condition [24]. Therefore, for simplicity, we below consider the cases of  $p = 2$ , i.e. one-matrix models. Then we can completely solve the conditions and the Stokes matrices are given with  $(L = 1, 2, \dots)$  as

1)  $r = q + 2 = 4L + 1$  cases ( $m = 1, 2, \dots, L$ )

$$\begin{aligned} S_{4m-5} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & S_{4m-3} &= \begin{pmatrix} 1 & \alpha_m \\ 0 & 1 \end{pmatrix} \\ S_{4L-1} &= \begin{pmatrix} 1 & 0 \\ \pm i & 1 \end{pmatrix}, & S_{4L+1} &= \begin{pmatrix} 1 & \pm i \\ 0 & 1 \end{pmatrix} \\ S_{4(2L-m)+3} &= \begin{pmatrix} 1 & 0 \\ \alpha_{-m} & 1 \end{pmatrix}, & S_{4(2L-m)+5} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

with  $\sum_{m=1}^L (\alpha_m + \alpha_{-m}) = \pm i$ .

<sup>2</sup> The constraint requires “ $p$  cuts” (not one cut) around  $\lambda \rightarrow \infty$  in the resolvent function of this  $p \times p$  isomonodromy systems. It is because the spectral parameter  $\lambda (= \zeta^{1/p})$  creates  $p$  copies of the physical cuts in  $\mathbb{Z}_p$ -symmetric way.





$$\mathcal{F}_{\text{asym}} \simeq \mathcal{F}_{\text{pert}}(g; \mu) + \mathcal{F}_{\text{nonpert.}}(g; \mu),$$

$$\mathcal{F}_{\text{nonpert.}}(g; \mu) = \mp \frac{\sqrt{g/2} \exp\left[+\frac{10}{21g} \sqrt{2(5-\sqrt{5})} \mu^{\frac{7}{4}}\right]}{\sqrt{5\pi(2\sqrt{5})^{\frac{3}{2}}(\sqrt{5}+1)^{\frac{5}{2}} \mu^{\frac{7}{8}}}} + \dots, \quad (13)$$

with  $u = g^2 \partial_t^2 \mathcal{F}(t; \mu)$ ,  $0 = 2t + 2(g/2)^4 u'''' - 5\mu u + 5(gu'/2)^2 + 10(g/2)^2 u u'' + 5u^3$  and  $t \rightarrow 0$ . This is the standard contribution from the  $(1, 2)$  *ghost* ZZ-brane (See also [15, 37]).<sup>3</sup> Note that  $\mathcal{F}(g)$  is a real function and here is shown the leading of one-instanton contributions, and the multi-instanton contributions have stronger exponential behavior. By this, we conclude that the Yang-Lee edge perturbative string vacuum is unstable. More precisely, since there is no tachyon in its perturbative spectrum, this string theory vacuum is *meta-stable*.

Generally meta-stable vacua in quantum systems have an important characteristic by *decay rate*.<sup>4</sup> Here, with use of the network, i.e. path-integral degree of freedom in string theory, we calculate the decay rate by applying the prescription of [35]. The way is to deform the path/network to avoid the instability (now by discarding hermiticity condition with keeping the single-line condition as in Fig. 2-c):

$$\mathcal{F}(g; \mu) \xrightarrow{\text{deform.}} \mathcal{F}^{(\text{def})}(g; \mu) \underset{\text{asym}}{\simeq} \mathcal{F}_{\text{pert}}(g; \mu) + \mathcal{F}_{\text{nonpert.}}^{(\text{def})}. \quad (14)$$

Then its imaginary part is the *decay rate* of the Yang-Lee edge string vacuum ( $g, \mu > 0$ ):

$$\text{Im } \mathcal{F}_{\text{nonpert.}}^{(\text{def})} = \frac{\mp 1}{2} \frac{\sqrt{g/2} \exp\left[-\frac{10}{21g} \sqrt{2(5+\sqrt{5})} \mu^{\frac{7}{4}}\right]}{\sqrt{5\pi(2\sqrt{5})^{\frac{3}{2}}(\sqrt{5}-1)^{\frac{5}{2}} \mu^{\frac{7}{8}}}} + \dots. \quad (15)$$

This is *one half* of the standard  $(1, 1)$  ZZ-brane contribution (See also [37]).

Generally one can see that the  $(2, q)$  minimal string theory is meta-stable: The string theory with hermiticity has ghost  $(1, 2m)$  ZZ-branes ( $m = 1, 2, \dots, (q-3)/2$ ) which render the vacuum unstable; the decay rate is given by half of the  $(1, 1)$  ZZ-brane. String theory of pure-gravity,  $(2, 3)$ , is an exception, since there is no ghost ZZ-brane in its background and therefore it is a stable

vacuum. In this way, we have shown that physics of the landscape is given by the networks/fugacity which control the spectrum of (ghost) D-instantons. Importantly, this example shows that different networks (Fig. 2-b1 and 2-c), i.e. different fugacities, may represent different physical situations of the same theory.

## B. The true vacuum and landscape of string theory

Since the Yang-Lee edge string theory is meta-stable, there is *the true vacuum*, into which the string theory decays. For that purpose, we choose the spectral curve in the landscape,

$$\varphi_{\text{tv}}(\zeta) \in \mathfrak{L}_{\text{str}} \quad (\varphi_{\text{mstr}}(\zeta) \in \mathfrak{L}_{\text{str}}), \quad (16)$$

in such a way that there is large instantons along the Deift-Zhou network of the RH integral (8). Roughly speaking, the instanton actions on the saddle points  $\lambda_*$  ( $\partial_\lambda [\varphi^{(j)}(\lambda_*) - \varphi^{(l)}(\lambda_*)] = 0$ ) should be positive real:

$$\text{Re}[\varphi^{(j)}(\zeta_*) - \varphi^{(l)}(\zeta_*)] < 0, \quad (17)$$

if the corresponding lines of the network  $\mathcal{K}_m$  (with non-zero Stokes multiplier  $s_{m,j,l} \neq 0$ ) are attached to the saddle point  $\lambda^*$ . In the current cases of  $p = 2$ , it is eventually equivalent to vanishing condition around B-cycle on the network  $\mathcal{K}$ :

$$\oint_B d\zeta \partial_\zeta [\varphi_{\text{tv}}^{(1)}(\zeta) - \varphi_{\text{tv}}^{(2)}(\zeta)] = 0 \quad B \subset \mathcal{K}, \quad (18)$$

which is known as the Boutroux equations in the RH context [20]. This kind of condition has been discussed also in old literatures of matrix models [5]. This simply means that the eigenvalues should fill up to the same Fermi-level of the effective potential along the DZ network. Here we simply show the result:

$$\zeta = \sqrt{\mu}(\wp(z) + c), \quad \partial_\zeta \varphi_{\text{tv}}^{(1)}(\zeta^{1/2}) = \sqrt{2\mu^{\frac{5}{2}}}(\wp(z) - \alpha)\wp'(z). \quad (19)$$

Here the Weierstrass  $\wp$  function is given by  $(\wp'(z))^2 = 4(\wp(z))^3 - g_2\wp(z) - g_3$ . The normalization of the system is now fixed as  $\alpha = \frac{5}{2}c$ ,  $g_2 = 5(1-3c^2)$ ,  $g_3 = 5c(2-7c^2)$  so that the corresponding string-background  $\varphi_{\text{tr}}(\zeta)$  belongs to the landscape  $\mathfrak{L}_{\text{str}}$ . Therefore, the parameter  $c$  is *an coordinate of the string theory landscape* of the Yang-Lee edge, and the true-vacuum condition is expressed with the Weierstrass elliptic functions:

$$\left[\frac{4g_2}{5}\zeta_W(\omega_B) - \frac{6g_3\omega_B}{5}\right]\alpha = \frac{6g_3}{7}\zeta_W(\omega_B) - \frac{g_2^2\omega_B}{21}, \quad (20)$$

where  $\omega_B$  is the Weierstrass half period along the  $B$ -cycle, and  $\zeta_W(z)$  is the Weierstrass  $\zeta$  function,  $\zeta'_W(z) = -\wp(z)$ . The numerical value of  $c$  is given as  $c \simeq -0.184963725\dots$ . Then the perturbative amplitude

<sup>3</sup> For more about ZZ branes [38], See also [8].

<sup>4</sup> Here we define *decay rates* by the imaginary part of “energy” of the meta-stable states in the following sense:  $e^{\mathcal{F}} = \langle \text{vac} | e^{-TH} | \text{vac} \rangle \sim e^{-TE_{\text{vac}}}$  ( $T \rightarrow \infty$ ),  $E_{\text{vac}} = E + i\Gamma_{\text{vac}}$ . Since minimal string theory is an Euclidean theory with “compact Euclidian time”, our decay rate is simply given by imaginary part of the free-energy of meta-stable vacuum. Therefore, this definition/terminology can be easily generalized to the Lorenzian situations of string theory.

around this true vacuum is obtained (by the RH approach with the network of Fig. 2-b2) as

$$u(\mu) \simeq -\sqrt{\mu} \left( \wp(\omega_A) + \wp(\omega_B) - \wp(\omega_A + \omega_B) + c \right). \quad (21)$$

Here  $\omega_A$  is the half period of the  $A$ -cycle.

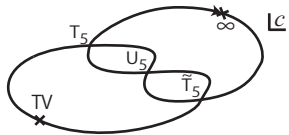


FIG. 3: One parameter landscape of Yang-Lee edge string theory. The coordinate is given by  $c$ .  $T_5$ ,  $\bar{T}_5$  and  $U_5$  are backgrounds which are described by spectral curves with genus-zero. The value of  $c$  for each vacuum is  $c = \frac{1 \pm \sqrt{5}}{6}, \pm \frac{1 + \sqrt{5}}{6}, \frac{\pm \sqrt{5}}{3\sqrt{2}}$ , respectively.

The perturbative structure around the true vacuum does not receive any contributions from non-perturbative ambiguities, which is a result of universality. Note that, since the expression includes elliptic functions, this vacuum represents non-perturbative vacuum whose classical dynamics would not be stringy degree of freedom although quantum corrections still resembles the closed-string behavior  $g^{2n-2}$ . It would be worth drawing the string theory landscape with the parameter  $c$  (Fig. 3). The pinched points correspond to perturbative-string

vacua ( $T_5$ ,  $\bar{T}_5$  and  $U_5$ ) in which the perturbative amplitudes have simple the power scale behavior.  $T_5$  is the original minimal-string vacuum (10). Note that not all the vacua have a simple interpretation by matrix models and therefore most of the vacua are off-shell background of this non-perturbative string theory.

These analyses can clearly be generalized to many other systems. Further investigations including general  $(p, q)$  cases will be reported in future communication [39].

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